JOURNAL OF ALGEBRA 1, 215-232 (1964)

S-Rings and the Irreducible Representations of Finite Groups

OLAF TAMASCHKE

*Mathematical Institute, 10 Parks Road, Oxford, England**

Communicated by Graham Higman

Received February 13, 1964

This paper proves the coefficients in the irreducible representations of a finite group $\mathfrak{G}$ to be a link between the irreducible representations of the S-rings on $\mathfrak{G}$ and their related class functions. The special case of commutative S-rings has been dealt with in [*6*], that of S-rings contained in the center of the group algebra can be found in [*4*] and [*5*]. An application of this theory to the orders of the centralizers of the elements in finite groups has been given in [*2*].

Let Γ be the group algebra of the finite group $\mathfrak{G}$ over the field **C** of complex numbers. A subalgebra T of Γ is said to be an *S-ring* on $\mathfrak{G}$ [*8*; *9*, 23.1] if there exists a decomposition

$$\mathfrak{G} = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_t$$

of the group $\mathfrak{G}$ into nonempty, trivially intersecting sets $\mathfrak{T}_i$ with the properties:

(1) The elements $\tau_i = \sum_{G \in \mathfrak{T}_i} G$ $(i = 1, \cdots, t)$ of Γ form a **C**-basis of T.

(2) For every $\mathfrak{T}_i$ there exists a $\mathfrak{T}_j$ consisting exactly of the inverses of all elements contained in $\mathfrak{T}_i$.

We call the sets $\mathfrak{T}_i$ the *T-classes* of $\mathfrak{G}$, and the τ_i the *simple basis elements* of T.

An S-ring T is called *unitary*, if the unit element E of $\mathfrak{G}$ is contained in T (and therefore forms a T-class by itself).

A complex-valued function on $\mathfrak{G}$ is said to be a *T-class function*, if it is constant on any T-class of $\mathfrak{G}$. We shall be concerned with the algebra $T^\#$ of all T-class functions which is provided with the algebraic compositions

$$(f+g)(G) = f(G) + g(G), \qquad (fg)(G) = f(G)\,g(G), \qquad (cf)(G) = cf(G)$$

* Permanent address: Mathematisches Institut der Universität Tübingen, Germany.

1

for all $f, g \in T^{\#}$, all $G \in \mathfrak{G}$, and all $c \in \mathbf{C}$. Any $\mathbf{C}$-basis

$$\beta_i = \frac{1}{|\mathfrak{G}|} \sum_{G \in \mathfrak{G}} b_i(G^{-1})\, G \qquad (i = 1, \cdots, t)$$

of an S-ring T yields a $\mathbf{C}$-basis

$$b_i : G \to b_i(G), \qquad G \in \mathfrak{G} \qquad (i = 1, \cdots, t)$$

of the algebra $T^{\#}$ of all T-class functions, and vice versa. But we are mainly interested in those $\mathbf{C}$-bases of $T^{\#}$ which are connected with the coefficients of the irreducible representations of T. The connecting link is given by the coefficients of the irreducible representations of $\mathfrak{G}$ itself (Theorem 1.1). Since the computation of the irreducible representations of a finite group is sometimes difficult, we give another method of obtaining these special T-class functions. This is done by reducing completely the image $R(T)$ of T by a regular representation $R : G \to R(G)$ of $\mathfrak{G}$ instead of reducing R itself (Theorem 1.2). We apply our theory to those special S-rings on G which have as their T-classes the double cosets $\mathfrak{L}G\mathfrak{L}$ of a subgroup $\mathfrak{L}$ of $\mathfrak{G}$ (Theorem 1.3).

The fact that the T-class functions related (in the sense of Theorem 1.1) to the irreducible representations of T can be expressed in terms of the coefficients of the irreducible representations of $\mathfrak{G}$ is however of theoretical importance. It gives rise to numerical relations between the irreducible representations of T and their related T-class functions (Section II). The most useful of those relations (Theorem 2.8) implies as a special case a well known theorem of Frame [*1*] and Wielandt [9, 30.5] which connects the degrees of the irreducible constituents of the permutation representation of any finite transitive permutation group $\mathfrak{G}$ with the lengths of the systems of transitivity of the subgroup $\mathfrak{G}_1$ of those elements of $\mathfrak{G}$ fixing one letter (Theorem 2.9). If we specialize T to be the center of the group algebra, then these numerical relations reduce to the well known orthogonality relations of the irreducible characters of finite groups.

I. T-Class Functions Related to the Irreducible Representations of T

Any S-ring T on a finite group $\mathfrak{G}$ is a semisimple algebra [*8*, p. 386, footnote; *9*, remark after 23.3]. Every representation of T over $\mathbf{C}$ is therefore completely reducible. Since T is a subalgebra of the group algebra Γ of $\mathfrak{G}$, every irreducible representation of T occurs as an irreducible constituent of at least one of the irreducible representations of $\mathfrak{G}$, if they are linearly extended onto Γ and then restricted to T. These are the very simple but fundamental facts for that what follows.

THEOREM 1.1. *Let T be an S-ring on a finite group $\mathfrak{G}$ with the T-classes $\mathfrak{T}_i$ and the simple basis elements τ_i $(i = 1, \cdots, t)$. Let*

$$F_\nu : \tau \to F_\nu(\tau) = (f_{\alpha\beta}^{(\nu)}(\tau))_{\alpha,\beta=1,\cdots,y_\nu}, \qquad \tau \in T;$$

$$y_\nu = \text{degree}\, F_\nu \qquad (\nu = 1, \cdots, r)$$

be a complete set of pairwise inequivalent irreducible representations of T over $\mathbf{C}$, and let Δ be a complete set of pairwise inequivalent irreducible representations

$$D_\rho : G \to D_\rho(G) = (d_{\kappa\lambda}^{(\rho)}(G))_{\kappa,\lambda=1,\cdots,x_\rho}, \qquad G \in \mathfrak{G};$$

$$x_\rho = \text{degree}\, D_\rho \qquad (\rho = 1, \cdots, n)$$

of $\mathfrak{G}$ over $\mathbf{C}$ such that all the representations

$$\tau \to D_\rho(\tau), \qquad \tau \in T \qquad (\rho = 1, \cdots, n)$$

of T are completely reduced with the representations F_ν of T as their irreducible constituents. The complex-valued functions

$$d_{\kappa\lambda}^{(\rho)} : G \to d_{\kappa\lambda}^{(\rho)}(G), \qquad G \in \mathfrak{G} \qquad (\kappa, \lambda = 1, \cdots, x_\rho; \rho = 1, \cdots, n)$$

shall be identified with their linear extensions onto the group algebra Γ of $\mathfrak{G}$. We denote by

$$\hat{d}_{\kappa\lambda}^{(\rho)} : \tau \to d_{\kappa\lambda}^{(\rho)}(\tau), \qquad \tau \in T,$$

the restriction of $d_{\kappa\lambda}^{(\rho)}$ to T, and by

$$f_{\alpha\beta}^{(\nu)} : \tau \to f_{\alpha\beta}^{(\nu)}(\tau), \qquad \tau \in T,$$

the complex-valued functions on T defined by the coefficients of F_ν. Then we have:

(a) *The mappings*

$$\varphi_{\alpha\beta}^{(\nu)} : G \to \varphi_{\alpha\beta}^{(\nu)}(G) = \sum_{\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\alpha\beta}^{(\nu)}} x_\rho d_{\kappa\lambda}^{(\rho)}(G), \qquad G \in \mathfrak{G}$$

$$(\alpha, \beta = 1, \cdots, y_\nu; \nu = 1, \cdots, r)$$

are a $\mathbf{C}$-basis of the algebra $T^\#$ of all T-class functions of $\mathfrak{G}$.

(b) *The elements*

$$\eta_{\alpha\beta}^{(\nu)} = \frac{1}{|\mathfrak{G}|} \sum_{G \in \mathfrak{G}} \varphi_{\alpha\beta}^{(\nu)}(G^{-1})\, G \qquad (\beta = 1, \cdots, y_\nu)$$

are a $\mathbf{C}$-basis of a minimal right ideal $\mathfrak{R}_\alpha^{(\nu)}$ of T $(\alpha = 1, \cdots, y_\nu; \nu = 1, \cdots, r)$ with the property that the right multiplications of $\mathfrak{R}_\alpha^{(\nu)}$ by the elements $\tau \in T$

yield, related to this basis, the irreducible representation F_ν of T. They are "matrix units" of T, which means

$$\eta_{\gamma\epsilon}^{(\mu)}\eta_{\alpha\beta}^{(\nu)} = \delta_{\mu\nu}\delta_{\epsilon\alpha}\eta_{\gamma\beta}^{(\nu)}$$

where $\delta_{\mu\nu}$ and $\delta_{\epsilon\alpha}$ are Kronecker symbols.

We call the $\varphi_{\alpha\beta}^{(\nu)}$ $(\alpha, \beta = 1, \cdots, y_\nu)$ the T-class functions of $\mathfrak{G}$ *related* to F_ν by Δ. Our next Theorem 1.2 shows that the $\varphi_{\alpha\beta}^{(\nu)}$ are in fact the same for any Δ. The reference to Δ can then be dropped. The elements

$$\eta_\nu = \sum_{\alpha=1}^{y_\nu} \eta_{\alpha\alpha}^{(\nu)} = \frac{1}{|\mathfrak{G}|} \sum_{G\in\mathfrak{G}} \psi_\nu(G^{-1})\, G \qquad (\nu = 1, \cdots, r)$$

are the primitive idempotents of the centre of T. Their coefficient functions

$$\psi_\nu = \sum_{\alpha=1}^{y_\nu} \varphi_{\alpha\alpha}^{(\nu)} \qquad (\nu = 1, \cdots, r)$$

remain unchanged, if we replace F_ν by an equivalent representation (they are therefore independent of Δ as well). We call the ψ_ν $(\nu = 1, \cdots, r)$ the *T-characters* of $\mathfrak{G}$.

Proof. We use the **C**-basis

$$\epsilon_{\kappa\lambda}^{(\rho)} = \frac{1}{|\mathfrak{G}|} \sum_{G\in\mathfrak{G}} x_\rho d_{\lambda\kappa}^{(\rho)}(G^{-1})\, G \qquad (\kappa, \lambda = 1, \cdots, x_\rho; \rho = 1, \cdots, n)$$

of Γ, and form the elements

$$\eta_{\alpha\beta}^{(\nu)} = \sum_{\hat{d}_{\lambda\kappa}^{(\rho)} = f_{\beta\alpha}^{(\nu)}} \epsilon_{\kappa\lambda}^{(\rho)} \qquad (\alpha, \beta = 1, \cdots, y_\nu; \nu = 1, \cdots, r).$$

We shall prove these elements to have the properties stated in the theorem. Because of the assumption on the D_ρ we get

$$\eta_{\alpha\beta}^{(\nu)}\tau_i = \sum_{\hat{d}_{\lambda\kappa}^{(\rho)} = f_{\beta\alpha}^{(\nu)}} \sum_{G\in\mathfrak{T}_i} \epsilon_{\kappa\lambda}^{(\rho)} G = \sum_{\hat{d}_{\lambda\kappa}^{(\rho)} = f_{\beta\alpha}^{(\nu)}} \sum_{G\in\mathfrak{T}_i} \sum_{\mu=1}^{x_\rho} d_{\lambda\mu}^{(\rho)}(G) \cdot \epsilon_{\kappa\mu}^{(\rho)}$$

$$= \sum_{\hat{d}_{\lambda\kappa}^{(\rho)} = f_{\beta\alpha}^{(\nu)}} \sum_{\mu=1}^{x_\rho} d_{\lambda\mu}^{(\rho)}(\tau_i) \cdot \epsilon_{\kappa\mu}^{(\rho)} = \sum_{\gamma=1}^{y_\nu} f_{\beta\gamma}^{(\nu)}(\tau_i) \Bigg(\sum_{\hat{d}_{\mu\kappa}^{(\rho)} = f_{\gamma\alpha}^{(\nu)}} \epsilon_{\kappa\mu}^{(\rho)} \Bigg)$$

$$= \sum_{\gamma=1}^{y_\nu} f_{\beta\gamma}^{(\nu)}(\tau_i) \cdot \eta_{\alpha\gamma}^{(\nu)} .$$

The right multiplications of the elements $\eta_{\alpha\beta}^{(\nu)}$ ($\beta = 1, \cdots, y_\nu$) by the elements $\tau \in T$ therefore induce the representation F_ν of T. If, on the other hand, there exists a $d_{\kappa\kappa}^{(\rho)}$ with $\hat{d}_{\kappa\kappa}^{(\rho)} = 0$, then we have also $\hat{d}_{\kappa\lambda}^{(\rho)} = 0$ and $\hat{d}_{\lambda\kappa}^{(\rho)} = 0$ for any $\lambda = 1, \cdots, x_\rho$ by the special form assumed for D_ρ. The same calculation as before then yields

$$\epsilon_{\kappa\kappa}^{(\rho)}\tau = 0 \qquad \text{for all} \qquad \tau \in T.$$

Every irreducible representation F_ν of T is in a one-to-one correspondence to a minimal two-sided ideal $\mathfrak{I}_\nu$ of T which, by the semi-simplicity of T, is isomorphic to the ring $M_{y_\nu}(\mathbf{C})$ of all $y_\nu \times y_\nu$-matrices over $\mathbf{C}$. Hence there exists a $\mathbf{C}$-basis $e_{\alpha\beta}^{(\nu)}(\alpha, \beta = 1, \cdots, y_\nu)$ of $\mathfrak{I}_\nu$ such that

$$F_\nu(e_{\alpha\beta}^{(\nu)}) = (\delta_{\alpha\gamma}\delta_{\beta\epsilon})_{\gamma,\epsilon=1,\cdots,y_\nu}$$

are the matrix units of $M_{y_\nu}(\mathbf{C})$. Since $\mathfrak{I}_\nu$ is in the kernel of $\mathfrak{F}_\mu$ for $\mu \neq \nu$, we have

$$F_\mu(e_{\alpha\beta}^{(\nu)}) = 0 \qquad \text{for all} \qquad \mu \neq \nu \; (\mu = 1, \cdots, r).$$

From these facts we get

$$\eta_{\gamma\epsilon}^{(\mu)} e_{\alpha\beta}^{(\nu)} = \delta_{\mu\nu} \sum_{\sigma=1}^{y_\nu} f_{\epsilon\sigma}^{(\nu)}(e_{\alpha\beta}^{(\nu)})\, \eta_{\gamma\sigma}^{(\nu)} = \delta_{\mu\nu}\delta_{\alpha\epsilon}\eta_{\gamma\beta}^{(\nu)}.$$

The unit element E of $\mathfrak{G}$ can be written in the form

$$E = \sum_{\mu=1}^{r} \sum_{\gamma=1}^{y_\mu} \eta_{\gamma\gamma}^{(\mu)} + \sum_{\hat{d}_{\kappa\kappa}^{(\rho)}=0} \epsilon_{\kappa\kappa}^{(\rho)}.$$

We therefore obtain

$$e_{\alpha\beta}^{(\nu)} = Ee_{\alpha\beta}^{(\nu)} = \sum_{\mu=1}^{r} \sum_{\gamma=1}^{y_\mu} \eta_{\gamma\gamma}^{(\mu)} e_{\alpha\beta}^{(\nu)} = \eta_{\alpha\beta}^{(\nu)} \qquad (\alpha, \beta = 1, \cdots, y_\nu;\ \nu = 1, \cdots, r).$$

The elements $\eta_{\alpha\beta}^{(\nu)}$ are hence a $\mathbf{C}$-basis of T. If we write

$$\eta_{\alpha\beta}^{(\nu)} = \frac{1}{|\mathfrak{G}|} \sum_{G\in\mathfrak{G}} \varphi_{\beta\alpha}^{(\nu)}(G^{-1})\, G = \frac{1}{|\mathfrak{G}|} \sum_{G\in\mathfrak{G}} \Bigg(\sum_{\hat{d}_{\lambda\kappa}^{(\rho)} = f_{\beta\alpha}^{(\nu)}} x_\rho d_{\lambda\kappa}^{(\rho)}(G^{-1}) \Bigg)\, G,$$

then (a) is proved, since the $\varphi_{\alpha\beta}^{(\nu)}$ are obviously a $\mathbf{C}$-basis of $T^{\#}$. By what we have already shown the

$$\mathfrak{R}_\alpha^{(\nu)} = \sum_{\beta=1}^{y_\nu} \mathbf{C}\eta_{\alpha\beta}^{(\nu)} \qquad (\alpha = 1, \cdots, y_\nu;\ \nu = 1, \cdots, r)$$

are indeed minimal right ideals of T satisfying (b).

Remark. The T-class $\mathfrak{T}_1$ containing the unit element E of $\mathfrak{G}$ is a subgroup $\mathfrak{L}$ of $\mathfrak{G}$ and

$$\epsilon = \frac{1}{|\mathfrak{L}|} \sum_{L \in \mathfrak{L}} L = \frac{1}{|\mathfrak{L}|} \cdot \tau_1$$

is the unit element of T [8, (1.6)]. Therefore $F_\nu(\epsilon) = E_{y_\nu}$ is the unit matrix for all $\nu = 1, \cdots, r$. Then all $D_\rho(\epsilon)$ are diagonal matrices with only 0's and 1's as coefficients. The number of 1's in the maindiagonal is exactly the multiplicity of the identity representation of $\mathfrak{L}$ in the restriction of D_ρ to $\mathfrak{L}$. If and only if T is non-unitary, that is $\mathfrak{L} \neq \langle E \rangle$, there exists at least one D_ρ such that the diagonal matrix $D_\rho(\epsilon)$ contains zeros in the main diagonal, which means

$$\hat{d}_{\kappa\kappa}^{(\rho)} = 0 \qquad \text{for some} \qquad \kappa = 1, \cdots, x_\rho \,.$$

To have the T-class functions related to the irreducible representations of an S-ring T in terms of the coefficients of the irreducible group representations is very valuable for theoretical purposes as can be seen in Section II. We are now going to give another presentation of them which is better adapted to practical calculations. Nevertheless this presentation has an important theoretical aspect too: It proves the T-class functions related to an irreducible representation of T to be the same for any system Δ of irreducible group representations.

THEOREM 1.2. *Let T be an S-ring on the finite group $\mathfrak{G}$. Let $R : G \to R(G)$ be a regular representation of $\mathfrak{G}$ such that*

$$\tau \to R(\tau) = \begin{pmatrix} E_{z_1} \times F_1(\tau) \cdots 0 & & 0 \\ & \ddots & \\ 0 \cdots & E_{z_r} \times F_r(\tau) & 0 \\ 0 \cdots 0 & & 0 \end{pmatrix}, \qquad \tau \in T,$$

is completely reduced. (A zero matrix occurs in the lower right corner if and only if T is nonunitary.) We decompose conformably the matrices

$$R(G) = \begin{pmatrix} R_{11}(G) \cdots R_{1r}(G) & R_{1,r+1}(G) \\ \vdots & \\ R_{r1}(G) \cdots R_{rr}(G) & R_{r,r+1}(G) \\ R_{r+1,1}(G) \cdots R_{r+1,r}(G) & R_{r+1,r+1}(G) \end{pmatrix}, \qquad G \in \mathfrak{G},$$

$$R_{\nu\nu}(G) = \begin{pmatrix} S_{11}^{(\nu)}(G) \cdots S_{1z_\nu}^{(\nu)}(G) \\ \vdots \\ S_{z_\nu 1}^{(\nu)}(G) \cdots S_{z_\nu z_\nu}^{(\nu)}(G) \end{pmatrix} \qquad (\nu = 1, \cdots, r),$$

where $S_{ii}^{(\nu)}(\tau) = F_\nu(\tau)$, $\tau \in T$ $(i = 1, \cdots, z_\nu; \nu = 1, \cdots, r)$.

Let $\varphi_{\alpha\beta}^{(\nu)}$ $(\alpha, \beta = 1, \cdots, y_\nu; \nu = 1, \cdots, r)$ *be the T-class functions on* $\mathfrak{G}$ *related to the irreducible representations* F_ν *of* T *by a complete system* Δ *of irreducible representations of* $\mathfrak{G}$ *as in Theorem 1.1. Then we have:*

(a) $$(\varphi_{\alpha\beta}^{(\nu)}(G))_{\alpha,\beta=1,\cdots,y_\nu} = \sum_{i=1}^{z_\nu} S_{ii}^{(\nu)}(G) \qquad \textit{for all } G \in \mathfrak{G} \qquad (\nu = 1, \cdots, r).$$

(b) $$z_\nu = \varphi_{\alpha\alpha}^{(\nu)}(E) = \sum_{\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\alpha\beta}^{(\nu)}} x_\rho \textit{ for all } \alpha, \beta = 1, \cdots, y_\nu \textit{ and all } \nu = 1, \cdots, r.$$

The $\varphi_{\alpha\beta}^{(\nu)}$ are therefore the same for any Δ; we can refer to them merely as the T-class functions of $\mathfrak{G}$ *related* to F_ν. We call the multiplicity z_ν of F_ν in $R(T)$ the *degree* of $\varphi_{\alpha\beta}^{(\nu)}$ for all $\alpha, \beta = 1, \cdots, y_\nu$.

Proof. In order to make a regular representation $\tilde{R}$ of $\mathfrak{G}$ out of the $D_\rho \in \Delta$ we have to reproduce every D_ρ exactly x_ρ times (x_ρ = degree D_ρ) along the main diagonal. If we sum up all those coefficients of $\tilde{R}$ for which we have $\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\alpha\beta}^{(\nu)}$, we get the T-class functions related to F_ν by Δ:

$$\varphi_{\alpha\beta}^{(\nu)} = \sum_{\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\alpha\beta}^{(\nu)}} x_\rho d_{\kappa\lambda}^{(\rho)} .$$

These coefficients are of course in the (α, β)-positions of the irreducible constituents F_ν in $\tilde{R}(T)$. These two facts just mentioned are not altered, if we transform $\tilde{R}$ by a permutation matrix into a representation $\hat{R}$ such that merely the irreducible constituents of $\tilde{R}(T)$ are rearranged into the same arrangement as in $R(T)$:

$$\hat{R}(\tau) = R(\tau) = \begin{pmatrix} E_{z_1} \times F_1(\tau) \cdots 0 & & 0 \\ & \ddots & \\ 0 \cdots E_{z_r} \times F_r(\tau) & & 0 \\ 0 \cdots 0 & & 0 \end{pmatrix} \qquad \text{for all } \tau \in T.$$

$\hat{R}$ then satisfies the statements of our theorem. But there exists a non-singular Matrix A such that

$$R(G) = A^{-1}\hat{R}(G)\, A \qquad \text{for all} \qquad G \in \mathfrak{G}.$$

From this it follows

$$AR(\tau) = R(\tau)\, A \qquad \text{for all} \qquad \tau \in T,$$

which means

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_{r+1} \end{pmatrix}, \qquad A_\nu = B_\nu \times E_{y_\nu}, \qquad \text{degree } B_\nu = z_\nu \, (\nu = 1, \cdots, r)\,.$$

With $B_\nu^{-1} = (b_{ij}^*)_{i,j}$ we now have for $\nu = 1, \cdots, r$

$$R_{\nu\nu}(G) = A_\nu^{-1}\hat{R}_{\nu\nu}(G)\, A_\nu = (B_\nu^{-1} \times E_{y_\nu}) \begin{pmatrix} \hat{S}_{11}^{(\nu)}(G) & \cdots & \hat{S}_{1z_\nu}^{(\nu)}(G) \\ \vdots & & \\ \hat{S}_{z_\nu 1}^{(\nu)}(G) & \cdots & \hat{S}_{z_\nu z_\nu}^{(\nu)}(G) \end{pmatrix} (B_\nu \times E_{y_\nu}),$$

$$S_{ii}^{(\nu)}(G) = \sum_{j,k=1}^{z_\nu} b_{ij}^* \hat{S}_{jk}^{(\nu)}(G)\, b_{ki}\,,$$

$$\sum_{i=1}^{z_\nu} S_{ii}^{(\nu)}(G) = \sum_{j,k=1}^{z_\nu} \left(\sum_{i=1}^{z_\nu} b_{ki} b_{ij}^*\right) \hat{S}_{jk}^{(\nu)}(G) = \sum_{j=1}^{z_\nu} \hat{S}_{jj}^{(\nu)}(G) = (\varphi_{\alpha\beta}^{(\nu)}(G))_{\alpha,\beta}\,.$$

This is our statement (a); (b) is an immediate consequence of (a).

Let T be an S-ring on the finite group $\mathfrak{G}$. The T-class $\mathfrak{T}_1$ containing the unit element E of $\mathfrak{G}$ is a subgroup $\mathfrak{L}$ of $\mathfrak{G}$,

$$\epsilon = \frac{1}{|\mathfrak{L}|} \sum_{L \in \mathfrak{L}} L$$

is the unit element of T, and every T-class

$$\mathfrak{T}_i = \bigcup_{G \in \mathfrak{T}_i} \mathfrak{L}G\mathfrak{L} \qquad (i = 1, \cdots, t)$$

is the union of double cosets $\mathfrak{L}G\mathfrak{L}$ of $\mathfrak{G}$ [*8*, (1.6)]. But the elements

$$\sum_{H \in \mathfrak{L}G\mathfrak{L}} H, \qquad G \in \mathfrak{G},$$

are obviously the simple basis elements of an S-ring on $\mathfrak{G}$, which can be written (with Γ as the group algebra of $\mathfrak{G}$ on $\mathbf{C}$) as $\epsilon\Gamma\epsilon$. Let us apply our theory to the S-ring $\epsilon\Gamma\epsilon$.

The elements ϵG, $G \in \mathfrak{G}$, form a $\mathbf{C}$-basis of the right ideal $\epsilon\Gamma$ of Γ. Related to this basis $\mathfrak{G}$ is represented on the right-$\mathfrak{G}$-module $\epsilon\Gamma$ by the representation $D: G \to D(G)$ of $\mathfrak{G}$, which is induced by the identical representation of $\mathfrak{L}$.

The left multiplications of $\epsilon\Gamma$ by the elements of $\epsilon\Gamma\epsilon$ yield all the $\mathfrak{G}$-endomorphisms of $\epsilon\Gamma$ [3, (1.7)]. Related to the basis ϵG, $G \in \mathfrak{G}$, they give all the matrices permutable with the representation D of $\mathfrak{G}$. Let $D_1, \cdots, D_r$ be the irreducible constituents of D, and let e_ρ be the multiplicity of D_ρ in D. If we completely reduce the algebra $V(D)$ of all matrices permutable with D, it has exactly r irreducible constituents, the e_ρ's are their degrees, and the $x_\rho =$ degree D_ρ ($\rho = 1, \cdots, r$) are their multiplicities. Since $\epsilon\Gamma\epsilon$ is contained in $\epsilon\Gamma$, the regular representation of $\epsilon\Gamma\epsilon$ appears as a constituent of $V(D)$. The reduction of the algebra $V(D)$ has therefore in fact all the irreducible representations F_ν of $\epsilon\Gamma\epsilon$ as its irreducible constituents. Hence $\epsilon\Gamma\epsilon$ has r irreducible representations F_ν, and the degree y_ν of F_ν is equal to the multiplicity e_ρ of one of the irreducible constituents D_ρ of D.

It is a well known fact, that any irreducible representation D_ρ of $\mathfrak{G}$ appears in D as often as the identical representation of $\mathfrak{L}$ appears in the restriction of D_ρ to $\mathfrak{L}$. But this means $D_\rho(\epsilon\Gamma\epsilon) \neq 0$ if and only if D_ρ has a nonzero multiplicity e_ρ in D. In other words: all irreducible representations D_ρ of $\mathfrak{G}$ used in Theorem 1.1 in order to express the related $\epsilon\Gamma\epsilon$-class functions $\varphi_{\alpha\beta}^{(\nu)}$ occur in D, and, up to equivalence, only these. If we take any complete set of pairwise inequivalent irreducible representations F_ν ($\nu = 1, \cdots, r$) of $\epsilon\Gamma\epsilon$, then we can assume the $D_\rho(\epsilon\Gamma\epsilon)$ to be completely reduced with the F_ν as their irreducible constituents.

Any element of $\epsilon\Gamma$ has the form

$$\epsilon\gamma = \epsilon\gamma\epsilon + (\epsilon\gamma - \epsilon\gamma\epsilon), \qquad \gamma \in \Gamma,$$

with the first term in $\epsilon\Gamma\epsilon$, and the second term contained in the left annihilator $(\epsilon\Gamma\epsilon)_0$ of $\epsilon\Gamma\epsilon$ in $\epsilon\Gamma$. It follows that $\epsilon\Gamma$ as a right $\epsilon\Gamma\epsilon$ module is the direct sum

$$\epsilon\Gamma = \epsilon\Gamma\epsilon \oplus (\epsilon\Gamma\epsilon)_0,$$

and therefore contains exactly the regular representation of $\epsilon\Gamma\epsilon$, which is of degree $t = \sum_{\nu=1}^{r} y_\nu^2$, $y_\nu =$ degree F_ν, and the zero representation of degree $n - t$, $n = |\mathfrak{G} : \mathfrak{L}|$. Hence every irreducible representation F_ν of $\epsilon\Gamma\epsilon$ appears as an irreducible constituent of $D(\epsilon\Gamma\epsilon)$ with the multiplicity $y_\nu =$ degree F_ν. But the representation D of $\mathfrak{G}$ has altogether $\sum_{\rho=1}^{r} e_\rho = \sum_{\nu=1}^{r} y_\nu$ irreducible constituents D_ρ. Any $D_\rho(\epsilon\Gamma\epsilon)$ ($\rho = 1, \cdots, r$) contains at least one F_ν as an irreducible constituent. Hence any $D_\rho(\epsilon\Gamma\epsilon)$ ($\rho = 1, \cdots, r$) contains exactly one F_ν, and we can choose the notation such that

$$D_\nu(\epsilon\Gamma\epsilon) = \begin{pmatrix} F_\nu & 0 \\ 0 & 0 \end{pmatrix} \qquad (\nu = 1, \cdots, r).$$

Since F_ν cannot be contained in any $D_\rho(\epsilon\Gamma\epsilon)$, $\rho \neq \nu$, the $\epsilon\Gamma\epsilon$-class functions related to F_ν are $\varphi_{\alpha\beta}^{(\nu)} = x_\nu d_{\alpha\beta}^{(\nu)}$ ($\alpha, \beta = 1, \cdots, y_\nu$; $\nu = 1, \cdots, r$), and the degree z_ν of $\varphi_{\alpha\beta}^{(\nu)}$ is equal to the degree x_ν of D_ν. We collect the results just obtained.

THEOREM 1.3. *Let $\mathfrak{G}$ be a finite group, and let $\mathfrak{L}$ be a subgroup of $\mathfrak{G}$. The subalgebra*

$$\epsilon\Gamma\epsilon, \qquad \epsilon = \frac{1}{|\mathfrak{L}|}\sum_{L\in\mathfrak{L}} L,$$

of the group algebra Γ of $\mathfrak{G}$ over $\mathbf{C}$ is an S-ring on $\mathfrak{G}$ with the double cosets $\mathfrak{L}G\mathfrak{L}$, $G \in \mathfrak{G}$, as the $\epsilon\Gamma\epsilon$-classes of $\mathfrak{G}$. Let F_ν $(\nu = 1, \cdots, r)$ be a complete set of pairwise inequivalent irreducible representations of $\epsilon\Gamma\epsilon$ over $\mathbf{C}$. Denote by $D : G \to D(G)$ the representation of $\mathfrak{G}$, which is induced by the identical representation of $\mathfrak{L}$. Take the irreducible constituents $D_1, \cdots, D_s$ of D such that each $D_\sigma(\epsilon\Gamma\epsilon)$ is completely reduced with the F_ν as its irreducible constituents. Then we have (with an appropriate numeration of the D_σ and the F_ν):

(a) $r = s$.

(b) *For any irreducible representation D_ρ of $\mathfrak{G}$ it is $D_\rho(\epsilon\Gamma\epsilon) \neq 0$ if and only if D_ρ is a constituent of D.*

(c) $D_\nu(\epsilon\Gamma\epsilon) = \begin{pmatrix} F_\nu & 0 \\ 0 & 0 \end{pmatrix}$ *for all* $\nu = 1, \cdots, r$.

(d) *The multiplicity e_ν of D_ν in D is equal to the degree y_ν of F_ν $(\nu = 1, \cdots, r)$.*

(e) *The complex-valued functions*

$$\varphi_{\alpha\beta}^{(\nu)} : G \to \varphi_{\alpha\beta}^{(\nu)}(G) = x_\nu d_{\alpha\beta}^{(\nu)}(G), \qquad G \in \mathfrak{G} \qquad (\alpha, \beta = 1, \cdots, y_\nu),$$

defined by the coefficients and the degree x_ν of

$$D_\nu(G) = (d_{\kappa\lambda}^{(\nu)}(G))_{\kappa,\lambda=1,\cdots,x_\nu}$$

are the $\epsilon\Gamma\epsilon$-class functions of $\mathfrak{G}$ related to F_ν $(\nu = 1, \cdots, r)$. This means: The functions

$$d_{\alpha\beta}^{(\nu)} : G \to d_{\alpha\beta}^{(\ )}(G), \qquad G \in \mathfrak{G} \qquad (\alpha, \beta = 1, \cdots, y_\nu;\ \nu = 1, \cdots, r)$$

are a $\mathbf{C}$-basis of the algebra of all complex-valued functions on $\mathfrak{G}$ which are constant on every double coset $\mathfrak{L}G\mathfrak{L}$, $G \in \mathfrak{G}$.

(f) *The degree $z_\nu = \varphi_{\alpha\alpha}^{(\nu)}(E)$ of $\varphi_{\alpha\beta}^{(\nu)}$ $(\alpha, \beta = 1, \cdots, y_\nu)$ is equal to the degree x_ν of D_ν $(\nu = 1, \cdots, r)$.*

The results on the special S-rings $\epsilon\Gamma\epsilon$ are translatable into the language of permutation groups. If $\mathfrak{G}$ is a finite transitive permutation group, then take $\mathfrak{L}$ to be the subgroup $\mathfrak{G}_1$ of all elements of $\mathfrak{G}$ fixing one letter. The cosets $\mathfrak{G}_1 G \mathfrak{G}_1$ are in a one-to-one correspondence with the systems of transitivity of $\mathfrak{G}_1$, and D is the permutation representation of $\mathfrak{G}$.

As a matter of fact the notion of an S-ring had its origin in the theory of finite permutation groups [9, Chap. IV]. It appeared first in the study of finite permutation groups having a regular subgroup $\mathfrak{H}$. We then have $\mathfrak{G} = \mathfrak{G}_1\mathfrak{H}$, $\mathfrak{G}_1 \cap \mathfrak{H} = \langle 1 \rangle$. The "module of transitivity" $\mathbf{C}(\mathfrak{H}, \mathfrak{G}_1)$ [9, §21], which is a unitary S-ring on $\mathfrak{H}$ [9, 24.1], is isomorphic to $\epsilon\Gamma\epsilon$ in the strong sense, that every simple basis element of $\mathbf{C}(\mathfrak{H}, \mathfrak{G}_1)$ is mapped onto a simple basis element times $1/|\mathfrak{G}_1|$ of $\epsilon\Gamma\epsilon$ [3, p. 398]. Investigating $\mathbf{C}(\mathfrak{H}, \mathfrak{G}_1)$ instead of dealing with $\epsilon\Gamma\epsilon$ has in this case the advantage of being concerned with a subgroup of $\mathfrak{G}$ only instead of $\mathfrak{G}$ itself.

II. Numerical Relations

In this section T always denotes an S-ring on a finite group $\mathfrak{G}$ of order g with the T-classes $\mathfrak{T}_i$ and the simple basis elements τ_i $(i = 1, \cdots, t)$. We take a complete set F_ν $(\nu = 1, \cdots, r)$ of pairwise inequivalent irreducible representations of T over $\mathbf{C}$, and a set Δ of pairwise inequivalent irreducible representations D_ρ $(\rho = 1, \cdots, n)$ of $\mathfrak{G}$ such that the representations $\tau \rightarrow D_\rho(\tau)$ of T are completely reduced with the F_ν as their irreducible constituents. Since T is a semisimple algebra, we have

$$t = \dim_{\mathbf{C}} T = \sum_{\nu=1}^{r} y_\nu^2 , \qquad y_\nu = \text{degree}\, F_\nu .$$

We use the notation $\tau_i^* = \sum_{G \in T_i} G^{-1}$ and form the square matrices

$$F = (f_{\alpha\beta}^{(\nu)}(\tau_i))_{(\nu,\alpha,\beta,)i} , \qquad F^* = (f_{\beta\alpha}^{(\nu)}(\tau_i^*))_{(\nu,\alpha,\beta),i} ,$$

where the triples (ν, α, β) with $\alpha, \beta = 1, \cdots, y_\nu$; $\nu = 1, \cdots, r$ in their lexicographic order are used as row indices, and the $i = 1, \cdots, t$ as column indices. We shall always denote by G_i an element of $\mathfrak{T}_i$. With the T-class functions $\varphi_{\alpha\beta}^{(\nu)}$ related to the irreducible representations F_ν of T we form the square matrices

$$\Phi = (\varphi_{\alpha\beta}^{(\nu)}(G_i))_{(\nu,\alpha,\beta),i} , \qquad \Phi^* = (\varphi_{\beta\alpha}^{(\nu)}(G_i^{-1}))_{(\nu,\alpha,\beta),i} .$$

Finally we form diagonal matrices out of the lengths $t_i = |\mathfrak{T}_i|$ $(i = 1, \cdots, t)$ of the T-classes $\mathfrak{T}_i$, and of the degrees

$$z_\nu = \sum_{d_{\kappa\lambda}^{(\rho)} = f_{\alpha\beta}^{(\nu)}} x_\rho$$

of the $\varphi_{\alpha\beta}^{(\nu)}$ $(\alpha, \beta = 1, \cdots, y_\nu; \nu = 1, \cdots, r)$.

$$\mathrm{T} = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_t \end{pmatrix}, \qquad Z = \begin{pmatrix} z_1 \cdot E_{y_1^2} & & 0 \\ & \ddots & \\ 0 & & z_r \cdot E_{y_r^2} \end{pmatrix}.$$

We shall repeatedly use the presentation of the $\varphi_{\alpha\beta}^{(\nu)}$ in terms of the coefficients of Δ given by Theorem 1.1.

PROPOSITION 2.1. $ZF = \Phi\mathrm{T}$.

Proof. Taking Theorem 1.1 into account we get

$$z_\nu f_{\alpha\beta}^{(\nu)}(\tau_i) = \sum_{\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\alpha\beta}^{(\nu)}} x_\rho d_{\kappa\lambda}^{(\rho)}(\tau_i) = \sum_{G \in \mathfrak{T}_i} \sum_{\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\alpha\beta}^{(\nu)}} x_\rho d_{\kappa\lambda}^{(\rho)}(G) = \sum_{G \in \mathfrak{T}_i} \varphi_{\alpha\beta}^{(\nu)}(G)$$

$$= \varphi_{\alpha\beta}^{(\nu)}(G_i)\, t_i\,.$$

We always denote by A' the transposed of any matrix A. Then we have:

PROPOSITION 2.2. $F^*\Phi' = g \cdot E_t$.

Here g is the order of $\mathfrak{G}$, and E_t means the unit matrix of degree t. In the special case of $T = Z =$ center of the group algebra Γ, these relations are the orthogonality relations of the irreducible characters written in a slightly different form from usual.

Proof. Applying Theorem 1.1 we calculate the $((\nu, \alpha, \beta), (\mu, \gamma, \epsilon))$-coefficient of the left side. For this purpose we take a $d_{\kappa\lambda}^{(\rho)}$ with $\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\beta\alpha}^{(\rho)}$.

$$\sum_{i=1}^{t} f_{\beta\alpha}^{(\nu)}(\tau_i^*)\, \varphi_{\gamma\epsilon}^{(\mu)}(G_i) = \sum_{\hat{d}_{\eta\vartheta}^{(\sigma)} = f_{\gamma\epsilon}^{(\mu)}} \sum_{i=1}^{t} \sum_{G \in \mathfrak{T}_i} x_\sigma d_{\kappa\lambda}^{(\rho)}(G^{-1})\, d_{\eta\vartheta}^{(\sigma)}(G)$$

$$= \sum_{\hat{d}_{\eta\vartheta}^{(\sigma)} = f_{\gamma\epsilon}^{(\mu)}} x_\sigma \sum_{G \in \mathfrak{G}} d_{\kappa\lambda}^{(\rho)}(G^{-1})\, d_{\eta\vartheta}^{(\sigma)}(G) = g \sum_{\hat{d}_{\eta\vartheta}^{(\sigma)} = f_{\gamma\epsilon}^{(\mu)}} \delta_{\rho\sigma}\delta_{\kappa\vartheta}\delta_{\lambda\eta}$$

$$= \begin{cases} g & \text{if} \quad (\nu, \alpha, \beta) = (\mu, \gamma, \epsilon), \\ 0 & \text{if} \quad (\nu, \alpha, \beta) \neq (\mu, \gamma, \epsilon). \end{cases}$$

PROPOSITION 2.3. $F'ZF^* = g \cdot \mathrm{T}$.

For $T =$ center of Γ, this is once more equivalent to the orthogonality relations of the columns in the character table of $\mathfrak{G}$.

Proof. Using Theorem 1.1 the (i,j)-coefficient of the left side is

$$\sum_{\nu=1}^{r}\sum_{\alpha,\beta=1}^{y_\nu} f_{\alpha\beta}^{(\nu)}(\tau_i)\, z_\nu f_{\beta\alpha}^{(\nu)}(\tau_j^*) = \sum_{\nu=1}^{r}\sum_{\alpha=1}^{y_\nu} z_\nu f_{\alpha\alpha}^{(\nu)}(\tau_i\tau_j^*)$$

$$= \sum_{\nu=1}^{r}\sum_{\alpha=1}^{y_\nu}\sum_{\hat{d}_{\kappa\kappa}^{(\rho)}=f_{\alpha\alpha}^{(\nu)}} x_\rho d_{\kappa\kappa}^{(\rho)}(\tau_i\tau_j^*)$$

$$= \sum_{G\in\mathfrak{T}_i}\sum_{H\in\mathfrak{T}_j}\sum_{\rho=1}^{n}\sum_{\kappa=1}^{x_\rho} x_\rho d_{\kappa\kappa}^{(\rho)}(GH^{-1})$$

$$= g t_i \delta_{ij}\,.$$

Here we have to remember the remark after the proof of Theorem 1.1 that in the case of a nonunitary S-ring, and only in this case, there are some $d_{\kappa\kappa}^{(\rho)}$ annihilating T. Adding them in to get the character of the regular representation of $\mathfrak{G}$ does not affect the calculation.

PROPOSITION 2.4. $\Phi^*T\Phi' = g\cdot Z$.

This relation reduces to the orthogonality relations of the rows in the character table of $\mathfrak{G}$, if we specialize T to be the center of Γ.

Proof. Calculating the $((\nu,\alpha,\beta),(\mu,\gamma,\epsilon))$-coefficient of the left side we get

$$\sum_{i=1}^{l}\varphi_{\beta\alpha}^{(\nu)}(G_i^{-1})\, t_i \varphi_{\gamma\epsilon}^{(\mu)}(G_i) = \sum_{G\in\mathfrak{G}}\varphi_{\beta\alpha}^{(\nu)}(G^{-1})\,\varphi_{\gamma\epsilon}^{(\mu)}(G)$$

$$= \sum_{\hat{d}_{\kappa\lambda}^{(\rho)}=f_{\beta\alpha}^{(\nu)}}\ \sum_{\hat{d}_{\tau\omega}^{(\sigma)}=f_{\gamma\epsilon}^{(\mu)}} x_\rho x_\sigma \sum_{G\in\mathfrak{G}} d_{\kappa\lambda}^{(\rho)}(G^{-1})\, d_{\tau\omega}^{(\sigma)}(G)$$

$$= \sum_{\hat{d}_{\kappa\lambda}^{(\rho)}=f_{\beta\alpha}^{(\nu)}}\ \sum_{\hat{d}_{\tau\omega}^{(\sigma)}=f_{\gamma\epsilon}^{(\mu)}} g x_\rho \delta_{\rho\sigma}\delta_{\kappa\omega}\delta_{\lambda\tau}$$

$$= \begin{cases} g z_\nu & \text{if} \quad (\nu,\alpha,\beta) = (\mu,\gamma,\epsilon), \\ 0 & \text{if} \quad (\nu,\alpha,\beta) \neq (\mu,\gamma,\epsilon). \end{cases}$$

We introduce the matrix

$$F^{+} = (f_{\beta\alpha}^{(\nu)}(\tau_i))_{(\nu,\alpha,\beta),i}$$

which differs from F by replacing each coefficient $f_{\alpha\beta}^{(\nu)}(\tau_i)$ by $f_{\beta\alpha}^{(\nu)}(\tau_i)$ (instead of $f_{\beta\alpha}^{(\nu)}(\tau_i^*)$ as it is done in F^*).

PROPOSITION 2.5.

$$F'ZF^{+} = (gt_i\delta_{i^*j})_{i,j=1,\cdots,t}$$

where i^* *is defined by* $\tau_i^* = \tau_{i^*}$ $(i = 1, \cdots, t)$.

Proof. There are rational integers $a_{ijk} \geqslant 0$ such that

$$\tau_i\tau_j = \sum_{k=1}^{t} a_{ijk}\tau_k .$$

We calculate, as in the proof of Proposition 2.3, the (i, j)-coefficient of the left side.

$$\sum_{\nu=1}^{r} \sum_{\alpha,\beta=1}^{g_\nu} f_{\alpha\beta}^{(\nu)}(\tau_i)\, z_\nu f_{\beta\alpha}^{(\nu)}(\tau_j) = \sum_{\nu=1}^{r} \sum_{\alpha=1}^{g_\nu} z_\nu f_{\alpha\alpha}^{(\nu)}(\tau_i\tau_j)$$

$$= \sum_{k=1}^{t} a_{ijk} \sum_{\nu=1}^{r} \sum_{\alpha=1}^{g_\nu} \sum_{\hat{d}_{\kappa\kappa}^{(\rho)} = f_{\alpha\alpha}^{(\nu)}} x_\rho d_{\kappa\kappa}^{(\rho)}(\tau_k)$$

$$= \sum_{k=1}^{t} a_{ijk} \sum_{G\in\mathfrak{T}_k} \sum_{\rho=1}^{n} \sum_{\kappa=1}^{x_\rho} x_\rho d_{\kappa\kappa}^{(\rho)}(G).$$

If we assume the unit element E of $\mathfrak{G}$ to be contained in $\mathfrak{T}_1$, this is equal to a_{ij1}. Since

$$a_{ij1} = \begin{cases} t_i & \text{if} \quad \tau_i^* = \tau_j , \\ 0 & \text{otherwise,} \end{cases}$$

our proposition is proved.

The $\varphi_{\alpha\beta}^{(\nu)}$ are a $\mathbf{C}$-basis of T. There are therefore complex numbers

$$b\begin{pmatrix} \nu & \alpha & \beta \\ \mu & \gamma & \epsilon \\ \omega & \eta & \vartheta \end{pmatrix}$$

such that

$$\varphi_{\alpha\beta}^{(\nu)}\varphi_{\gamma\epsilon}^{(\mu)} = \sum_{(\omega,\eta,\vartheta)} b\begin{pmatrix} \nu & \alpha & \beta \\ \mu & \gamma & \epsilon \\ \omega & \eta & \vartheta \end{pmatrix} \varphi_{\eta\vartheta}^{(\omega)} .$$

Let $D_1 = \chi_1$ be the identical representation of G. We assume $\varphi_{11}^{(1)} = \chi_1$. With this notations we get:

PROPOSITION 2.6.

$$\Phi \mathrm{T} \Phi' = \left(gb \begin{pmatrix} \nu & \alpha & \beta \\ \mu & \gamma & \epsilon \\ 1 & 1 & 1 \end{pmatrix} \right)_{(\nu,\alpha,\beta),(\mu,\gamma,\epsilon)} .$$

Proof. The $((\nu, \alpha, \beta), (\mu, \gamma, \epsilon))$-coefficient of the left side is to be calculated as follows using Theorem 1.1.

$$\sum_{i=1}^{t} \varphi_{\alpha\beta}^{(\nu)}(G_i)\, t_i \varphi_{\gamma\epsilon}^{(\mu)}(G_i) = \sum_{G \in \mathfrak{G}} (\varphi_{\alpha\beta}^{(\nu)} \varphi_{\gamma\epsilon}^{(\mu)})(G)$$

$$= \sum_{(\omega,\eta,\vartheta)} b \begin{pmatrix} \nu & \alpha & \beta \\ \mu & \gamma & \epsilon \\ \omega & \eta & \vartheta \end{pmatrix} \sum_{\hat{d}_{\kappa\lambda}^{(\rho)} = f_{\eta\vartheta}^{(\omega)}} x_\rho \sum_{G \in \mathfrak{G}} d_{\kappa\lambda}^{(\rho)}(G)$$

$$= gb \begin{pmatrix} \nu & \alpha & \beta \\ \mu & \gamma & \epsilon \\ 1 & 1 & 1 \end{pmatrix}.$$

Proposition 2.3 can be used in order to express the coefficients a_{ijk} in the products

$$\tau_i \tau_j = \sum_{k=1}^{t} a_{ijk} \tau_k$$

in terms of the $f_{\alpha\beta}^{(\nu)}(\tau_i)$:

$$a_{ijk} = \frac{1}{gt_k} \sum_{\nu=1}^{r} z_\nu \sum_{\alpha,\beta,\gamma=1}^{y_\nu} f_{\alpha\beta}^{(\nu)}(\tau_i) f_{\beta\gamma}^{(\nu)}(\tau_j) f_{\gamma\alpha}^{(\nu)}(\tau_k^*) \qquad (i, j, k = 1, \cdots, t).$$

A similar expression holds for the

$$b \begin{pmatrix} \nu & \alpha & \beta \\ \mu & \gamma & \epsilon \\ \omega & \eta & \vartheta \end{pmatrix}$$

in terms of the $\varphi_{\alpha\beta}^{(\nu)}(G_i)$, using Proposition 2.4.

Let $\mathfrak{T}_1$ be the T-class of $\mathfrak{G}$ containing the unit element E of $\mathfrak{G}$. Since $\mathfrak{T}_1$ is a subgroup $\mathfrak{L}$ of $\mathfrak{G}$ [*8*, (*1.6*)], t_1 is a divisor of the group order g.

PROPOSITION 2.7. *det F is the square root of a rational integer; it is divisible by gt_1^{t-1}.*

Proof. Since F^* can be obtained from F by permuting some rows and some columns of F, we have $\det F^* = \pm \det F$. Proposition 2.3 yields

$$(\det F)^2 = \pm g^t \frac{\prod_{i=1}^{t} t_i}{\prod_{\nu=1}^{r} z_\nu^{y_\nu^2}}.$$

Every T-class $\mathfrak{T}_i$ is a union of double cosets $\mathfrak{T}_1 G \mathfrak{T}_1$ [*8*, (1.6)]. The integral coefficients $a_{ijk} \geqslant 0$ in

$$\tau_i \tau_j = \sum_{k=1}^{t} a_{ijk} \tau_k$$

as well as all the t_i are therefore divisible by t_1. Hence the elements

$$\sigma_i = \frac{1}{t_1} \tau_i \qquad (i = 1, \cdots, t)$$

form a $\mathbf{C}$-basis of T with the rational integers $b_{ijk} = a_{ijk}/t_1$ as the coefficients in their multiplication table. Then we get[1]

$$\sum_{\nu=1}^{r} \sum_{\alpha,\beta=1}^{y_\nu} f_{\alpha\beta}^{(\nu)}(\sigma_i) f_{\beta\alpha}^{(\nu)}(\sigma_j) = \sum_{\nu=1}^{r} \sum_{\alpha=1}^{y_\nu} f_{\alpha\alpha}^{(\nu)}(\sigma_i \sigma_j) = \sum_{k=1}^{t} b_{ijk} \sum_{\nu=1}^{r} \sum_{\alpha=1}^{y_\nu} f_{\alpha\alpha}^{(\nu)}(\sigma_k).$$

Let us denote $\epsilon = (1/t_1) \sum_{G \in \mathfrak{T}_1} G$ as in Theorem 1.3 with $\mathfrak{T}_1 \leqslant \mathfrak{G}$ instead of $\mathfrak{L}$. The left multiplication of $\epsilon\Gamma = \sum_{G\in\mathfrak{G}} \mathbf{C}\epsilon G$ by σ_k is represented, relative to the basis ϵG, $G \in \mathfrak{G}$, by a matrix with only zeros and ones as its coefficients [*3*, p. 397]. Since the representation of T by the left multiplication of $\epsilon\Gamma$ by the elements $\tau \in T$ contains the regular representation of T, this fact proves the traces $\sum_{\alpha=1}^{y_\nu} f_{\alpha\alpha}^{(\nu)}(\sigma_k)$ of all the matrices $F_\nu(\sigma_k)$ to be algebraic integers. If we define the matrices

$$C = (f_{\alpha\beta}^{(\nu)}(\sigma_i))_{(\nu,\alpha,\beta),i}, \qquad C^* = (f_{\beta\alpha}^{(\nu)}(\sigma_i))_{(\nu,\alpha,\beta),i},$$

then the above calculation yields the (i, j)-coefficient of $C'C^*$ which is an algebraic integer. Adding the jth column of $C'C^*$ to the first for all $j = 2, \cdots, t$, and taking $d_{\kappa\kappa}^{(\rho)}$ with $\hat{d}_{\kappa\kappa}^{(\rho)} = f_{\alpha\alpha}^{(\nu)}$ into account, we get

$$\sum_{\nu=1}^{r} \sum_{\alpha=1}^{y_\nu} f_{\alpha\alpha}^{(\nu)} \left(\frac{t_i}{t_1^2} \sum_{G\in\mathfrak{G}} G \right) = \frac{g t_i}{t_1^2}$$

[1] The author owes the following argument to Professor G. Higman.

as the coefficient in the ith row of the new first column. If we add afterwards the ith row to the first for $i = 2, \cdots, t$, we get in the same way gt_j/t_1^2 as the jth coefficient in the first row for $j = 2, \cdots, t$, and

$$\frac{g}{t_1^2}\sum_{i=1}^{t} t_i = \left(\frac{g}{t_1}\right)^2$$

as the (1,1)-coefficient. Our results show that

$$\left(\frac{t_1}{g}\cdot \det C\right)^2 = \pm\left(\frac{t_1}{g}\right)^2 \cdot \det (C'C^*)$$

is an algebraic integer. Hence the rational number

$$\left(\frac{1}{gt_1^{t-1}}\cdot \det F\right)^2 = \left(\frac{t_1}{g}\det C\right)^2$$

is a rational integer, which was to be proved.

The proof of Proposition 2.7 yields immediately:

THEOREM 2.8. *Let T be an S-ring on the finite group $\mathfrak{G}$ of order g with the T-classes $\mathfrak{T}_i$ of the lengths $t_i = |\mathfrak{T}_i|$ $(i = 1, \cdots, t)$, and with the unit element E of $\mathfrak{G}$ contained in $\mathfrak{T}_1$. Let y_ν $(\nu = 1, \cdots, r)$ be the degrees of the irreducible representations F_ν of T, and denote by z_ν the degree of the T-class functions on $\mathfrak{G}$ related to F_ν. Then we have:*

(a) *The number*

$$q = \left(\frac{g}{t_1}\right)^{t-2}\frac{\prod_{i=1}^{t}\frac{t_i}{t_1}}{\prod_{\nu=1}^{r} z_\nu^{y_\nu^2}}$$

is a rational integer.

(b) *If* $\det F$ *is a rational integer, then q is a square.*

We note that Proposition 2.7 and Theorem 2.8 are on the lines marked by Frame [*1*, Theorem B] and Wielandt [*9*, §30]. In fact if we apply our Theorem 2.8 to these special S-rings on $\mathfrak{G}$, which have as their T-classes the double cosets $\mathfrak{L}G\mathfrak{L}$ of a subgroup $\mathfrak{L}$ of $\mathfrak{G}$, then we get, taking Theorem 1.3 into account,

THEOREM OF FRAME AND WIELANDT 2.9. *Let $\mathfrak{G}$ be a finite transitive permutation group of degree n, and let $\mathfrak{G}_1$ be a subgroup of all those elements of $\mathfrak{G}$ fixing one letter. If $n_1, \cdots, n_t$ are the lengths of the systems of transitivity of $\mathfrak{G}_1$,* *and if the permutation representation D of $\mathfrak{G}$ has the irreducible representations $D_1, \cdots, D_r$ of $\mathfrak{G}$ as its irreducible constituents, each D_ρ with the multiplicity e_ρ, and if $x_\rho =$* degree *D_ρ $(\rho = 1, \cdots, r)$, then the number*

$$q = n^{t-2} \frac{\prod_{i=1}^{t} n_i}{\prod_{\rho=1}^{r} x_\rho^{e_\rho^2}}$$

is a rational integer.

If in a special case (for instance, if T is contained in the center of the group algebra [*4*, (2.17)]) det Φ is also an algebraic integer, then we can prove, using Proposition 2.4, that

$$q^{\#} = g^{t-2} \frac{\prod_{\nu=1}^{r} z_\nu^{y_\nu^2}}{\prod_{i=1}^{t} t_i}$$

is a rational integer too.

ACKNOWLEDGMENT

The author is very grateful to the British Council for a scholarship to the Mathematical Institute of Oxford University in the academic year 1963-64, where this paper was written.

REFERENCES

1. FRAME, J. S. The double cosets of a finite group. *Bull. Am. Math. Soc.* **47** (1941), 458-467.

2. FRAME, J. S. AND TAMASCHKE, O. Über die Ordnungen der Zentralisatoren der Elemente in endlichen Gruppen. *Math. Z.* **83** (1964), 41-45.

3. TAMASCHKE, O. Ringtheoretische Behandlung einfach transitiver Permutationsgruppen. *Math. Z.* **73** (1960), 393-408.

4. TAMASCHKE, O. Zur Theorie der Permutationsgruppen mit regulärer Untergruppe. I. *Math. Z.* **80** (1963), 328-354.

5. TAMASCHKE, O. Zur Theorie der Permutationsgruppen mit regulärer Untergruppe. II. *Math. Z.* **80** (1963), 443-465.

6. TAMASCHKE, O. *S*-Ringe und verallgemeinerte Charaktere auf endlichen Gruppen. *Math. Z.* **84** (1964), 101-119.

7. WIELANDT, H. Zur Theorie der einfach transitiven Permutationsgruppen. *Math. Z.* **40** (1935), 582-587.

8. WIELANDT, H. Zur Theorie der einfach transitiven Permutationsgruppen. II. *Math. Z.* **52** (1949), 384-393.

9. WIELANDT, H. Permutationsgruppen. Vorlesungen an der Universität Tübingen im Wintersemster 1954-55. Ausgearbeitet von J. André. English translation by R. Bercov, California Institute of Technology, Pasadena, Cal., 1962.